## zXtraThrill--Problem 15.33

This is a compound pendulum. That is, it is an extended object (versus a simple pendulum consisting of a single bob). If the upper rod's mass is assumed to be zero, what is the *period* of the system's oscillation?

This problem is interesting as it requires you to know how the governing equations for Simple Harmonic Motion were derived. If you remember back, our first brush with this kind of problem was with a spring. With the sign of the acceleration term left embedded, the derivation starting with N.S.L.:

$$\sum F_x :$$
$$-kx = ma_x$$
$$= m\frac{d^2x}{dt^2}$$
$$\Rightarrow \quad \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$
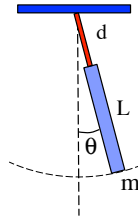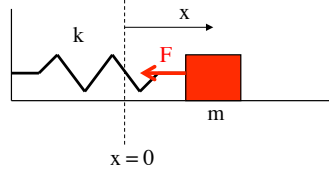
In other words, the solution $x = A\sin(\omega t + \phi)$ satisfied $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ *as long as* the motion's *angular frequency* is $\omega = \left(\frac{k}{m}\right)^{1/2}$.

The *angular frequency* and the *frequency* are related by $\omega = 2\pi\nu$, and the *frequency* and *period* are related by $T = 1/\nu$, so now you find yourself in an interesting situation. Specifically:

If you can sum the forces (or torques) that motivate a body to oscillate, and *if the resulting relationship can be put into the form*

$$1\frac{d^2x}{dt^2} + (K)x = 0 \quad \text{or} \quad 1\frac{d^2\theta}{dt^2} + (K)\theta = 0$$

where the "K" term is some constant that is related to parameters in the system, you know (because you've already done the math for a spring) that the system is oscillating with S.H.M. (these are the characteristic equations for S.H.M.), AND you know that the ANGULAR FREQUENCY ω will equal the square root of the "K" term in the equation (for the spring, the "K" term was "k/m" and, according to the math, ω was the square root of that).

The solution to $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ asks you to find a function "x" such that when you add the *second derivative of the function* to a *constant times the function*, it will always add to zero (that's what the expression says).

The function that satisfies this requirement is, in its most general form:

$$x = A\sin(\omega t + \phi).$$

If you put that position function back into the differential equation, you get:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$
$$\Rightarrow \quad \frac{d^2(A\sin(\omega t + \phi))}{dt^2} + \left(\frac{k}{m}\right)A\sin(\omega t + \phi) = 0$$
$$\Rightarrow \quad -\omega^2 A\sin(\omega t + \phi) + \left(\frac{k}{m}\right)A\sin(\omega t + \phi) = 0$$
$$\Rightarrow \quad \omega^2 = \left(\frac{k}{m}\right) \quad \Rightarrow \quad \omega = \left(\frac{k}{m}\right)^{1/2}$$

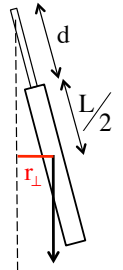
So how does that help us here?

Summing the torques about the pin yields:

$$\sum \Gamma_{pin} :$$
$$-mgL\sin\theta = I_{pin}\alpha$$

To finish we need the *moment of inertia* about the pin. We know it about the *center of mass* of a beam and we know that "d" in the sketch is equal to "L/2," so we can use the Parallel Axis Theorem and write:
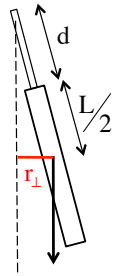
$$I_{pin} = I_{cm} + md^2$$
$$= \left(\frac{1}{12}mL^2\right) + m\left(\frac{L}{2} + d\right)^2$$
$$= \left(\frac{1}{12}mL^2\right) + m\left(\frac{L}{2} + \frac{L}{2}\right)^2$$
$$= \left(\frac{13}{12}mL^2\right)$$

With the appropriate moment of inertia, we can write:

$$\sum \Gamma_{pin}:$$

$$-mgL\sin\theta = I_{pin}\alpha$$

$$-mgL\sin\theta = \left(\frac{13}{12}mL^2\right)\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{12g}{13L}\sin\theta = 0$$

As you learned when first dealing with pendulum problems, this is not quite the right form. To get it into the right form, you have to do a small angle approximation, note that for small angles, $\sin\theta \Rightarrow \theta$ in radians, and with that we end up with:

$$\frac{d^2\theta}{dt^2} + \left(\frac{12g}{13L}\right)\theta = 0$$

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But great jumping hazzahs, this is the form for S.H.M. with a "K" term being "12g/13L." With that, the *angular frequency* must be:
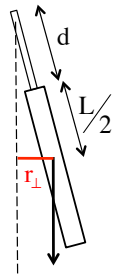
$$\omega = \left(\frac{12g}{13L}\right)^{1/2}$$

the frequency:

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left(\frac{12g}{13L}\right)^{1/2}$$

and the period:

$$T = \frac{1}{\nu} = 2\pi\left(\frac{13L}{12g}\right)^{1/2}$$

and aren't we clever!

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